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Aperture configuration optimality criterion for phased arrays of optical telescopes

Laurent M. Mugnier, Gérard Rousset, and Frédéric Cassaing

Office National d'Etudes et de Recherches Aérospatiales, Division Imagerie Optique à Haute Résolution, BP 72, F-92322 Châtillon cedex, France

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We address the optimization of the relative arrangement (aperture configuration) of a phased array of optical telescopes, coherently combined to form images of extended objects in a common focal plane. A novel optimality criterion, which is directly linked to the restoration error of the original object from the recorded image, is derived. This criterion is then refined into a second criterion to accommodate the possible knowledge of the noise spectrum. The optimal configuration is a function of the maximum spatial frequency of interest (or desired resolution) and takes into account the diameters of the elementary telescopes. Simulations illustrate the usefulness of this criterion for designing a synthetic-aperture optical instrument with three, four, and five telescopes.

Key words: synthetic aperture, phased arrays, aperture configuration, interferometry, optical imaging, image restoration.

1. INTRODUCTION

The relative arrangement of the elementary telescopes (the so-called aperture configuration, or pupil configuration) is a key aspect of the design of a synthetic-aperture instrument. There is an abundant literature on this subject in radio astronomy (see, in particular, the pioneering work of Moffet[1] and of Golay[2] and the papers by Cornwell[3] and by Lannes *et al.*[4]). More recently, many papers have discussed this subject with respect to optical instruments.[5–15]

The currently operating synthetic-aperture optics (SAO) instruments are two-aperture interferometers, which provide only visibility measurements[16]—although new instruments are under development for imaging purposes[17]—so that optimization of the aperture configuration is a relatively new topic in optics. Papers dealing with the aperture configuration optimization of a SAO instrument often use various criteria based on the shape of the point-spread function (PSF), such as the full width at half-maximum, the encircled energy, and the sidelobe level.[5–7,10,12] In these papers the best PSF is implicitly taken as that of the full-aperture telescope. Nevertheless, it has already been pointed out that the choice of an optimal aperture configuration should be based on Fourier domain considerations.[7]

In contrast, radio astronomers, because their data consist of sparse frequency plane samples of the object spectrum, have considered Fourier domain aperture optimization and have developed a number of data processing algorithms to obtain an estimate of the object. Since even very simple digital processing of the data (i.e., of the recorded image) can yield a better object estimate than the raw image itself, we believe that such data processing (i.e., an image restoration) should be done for an imaging SAO instrument. This image restoration can even be regarded as part of the observation system, the first part being the instrument itself. In the following, we assume that such processing is performed.

Some papers dealing with the aperture configuration optimization of an SAO system do take a quality criterion based on the uniform filling of the spatial-frequency plane[8,13] (the so-called $u-v$ plane) or on the maximization of the contiguous central core diameter of the optical transfer function,[14] (OTF) rather than on the shape of the PSF, but this uniformity is not very precisely defined. Also, the frequency coverage given by the elementary telescopes—which can be an advantage of optical wavelengths over radio wavelengths—is rarely[9,15] taken into account.

The importance of a compact configuration (i.e., one with no zeros in the spatial-frequency coverage) for imaging an extended object such as the Sun has already been stressed.[15] Indeed, when the object's support lies within the field of view, a constraint support can be used in the object estimation to recover frequencies that have not been recorded,[4] and, for a given desired resolution, the smaller the support, the more effective the support constraint.[18] For such objects, one can consider diluted configurations, still taking advantage of the frequency coverage of elementary telescopes. But this is not the case when the object (e.g., the Earth viewed from a satellite) extends over the whole field of view. A compact configuration is therefore a necessary condition for the imaging of extended objects without ambiguity, but this condition is not sufficient to determine the aperture configuration uniquely.

The purpose of this paper is to derive a criterion for aperture configuration optimization in the case of an instrument that images extended objects. The problem is to design the aperture array under external constraints such as the desired resolution (i.e., the maximum spatial frequency of interest), the total collecting surface (i.e., the signal-to-noise ratio requirement), and the system com-

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plexity (e.g., the number of elementary telescopes or the total size of the array).

In Section 2 a criterion is derived that minimizes the restoration error, i.e., the difference between the original object and the one estimated from the recorded image. This criterion, first presented in Ref. 19, defines rigorously what kind of frequency-plane uniformity is desirable to obtain an optimal configuration, and it explicitly takes into account the diameters of the elementary telescopes. In Section 3 this criterion is refined to accommodate the possible knowledge of the noise statistics. Then, in Section 4, computer- simulation results obtained with the defined criteria are presented.

2. APERTURE CONFIGURATION OPTIMALITY CRITERION

We consider a synthetic-aperture optical instrument that records images, that is, an instrument equivalent to a single telescope. This is in particular achieved with a phased array of elementary telescopes recombined homothetically[20] to form an image in a common focal plane. The recording process is modeled as

$$i = Ho + n, \tag{1}$$

where i is the recorded image, o is the original object, n is an additive noise, and H is the imaging operator in a Hilbert space $\mathbb{H}$ (e.g., the set of square integrable functions of two variables). The field-dependent aberrations are neglected in the following, so that the system is linear and shift invariant, and H is consequently a convolution operator of kernel h (the instrument's PSF):

$$i = h \star o + n. \tag{2}$$

If we let G be the restoration operator ($G = H^{-1}$, if H^{-1} exists, being the inverse filter), the estimated object reads as

$$o_e = Gi = GHo + Gn. \tag{3}$$

We define the restoration error by

$$\epsilon = \|o_e - o\| = \|(GH - I)o + Gn\|, \tag{4}$$

where I is the identity operator and $\|\cdot\|$ is the norm induced by the scalar product in $\mathbb{H}$.

We base our aperture configuration optimization on the minimization of the restoration error ϵ. Indeed, this error assesses the capability of the instrument (plus the restoration operator) to recover the object properly. We begin by deriving a bound on this error that is directly related to H and to G. With use of the triangular inequality,

$$\epsilon \leqslant \epsilon_o + \epsilon_n, \tag{5}$$

where

$$\epsilon_o = \|(GH - I)o\| \quad \text{and} \quad \epsilon_n = \|Gn\|. \tag{6}$$

The error term ϵ_o is a systematic type of error, which depends on the object o. It equals zero in particular if H is invertible and $G = H^{-1}$ and also if o belongs to the null space of $GH - I$ (that is, as we can see from the following, if GH is the identity up to the last frequencies of o). Nevertheless, in general, i.e., for an object o of infinite spectrum, H and G cannot be chosen so as to cancel ϵ_o. For a well-chosen G, ϵ_o is essentially due to the frequencies of the object above the cutoff of H; that is, ϵ_o is essentially determined by the choice of the instrument's resolution. In this paper we shall assume that the resolution (or, equivalently, the maximum frequency of interest) is already chosen by considerations regarding the types of object to be observed, and we shall optimize the configuration by minimizing the other term, ϵ_n, of the error.

This choice of resolution, which amounts to the choice of the frequency coverage of G, is similar to but different from the choice of the compromise between fidelity to the data (G close to H^{-1} and consequently ϵ_o small) and fidelity to the *a priori* information (smoothness of the solution, i.e., ϵ_n small), which is classical in ill-posed inverse problems.[21] Indeed, one should keep in mind that the present aim is not to best recover an object observed with a given instrument (which would involve the minimization of ϵ), but to *design* the instrument for a given resolution, so that our goal will be to minimize ϵ_n, the noise amplification that will occur during the restoration process, rather than ϵ. The relative noise amplification is defined by

$$\epsilon_n{}' = \frac{\|Gn\|}{\|o\|}. \tag{7}$$

Using the inequality $\|Ax\| \leqslant \|A\| \cdot \|x\|$, valid for any operator A and any vector x by definition of the norm of an operator, we see that

$$\epsilon_n \leqslant \|G\|\|n\| \tag{8}$$

$$\epsilon_n{}' \leqslant \|G\|\|H\| \frac{\|n\|}{\|Ho\|}. \tag{9}$$

It should be noted that the factor $\|n\|/\|Ho\|$ in Eq. (9) is the inverse of a signal-to-noise ratio (SNR) for the recorded image, since the L_2 norm is the square root of the integral of the spectral density of the signal. Likewise, $\epsilon_n{}'$ is the inverse of an SNR for the restored object. Thus the factor

$$c = \|G\|\|H\| \tag{10}$$

in relation (9) is a parameter that characterizes the degradation of the SNR (i.e., the noise amplification) during the imaging (H) plus restoration (G) process. It is the so-called condition number of numerical analysis when $G = H^{-1}$. It is this parameter c that will be used as a quality criterion for aperture configurations. Let us see now how to express c as a function of the OTF of the system.

First, the norm of an operator H is related to the eigenvalues of H^*H, where H^* is the adjoint of H. For a wide class of PSF's h (e.g., if h is square integrable), H is compact.[21] So H^*H is compact self-adjoint and, according to the Hilbert–Schmidt theorem (see, e.g., Ref. 22), has an eigenvalue decomposition. Additionally, H^*H is positive and $\|H^*H\| = \Lambda_s$, where Λ_s is the least upper bound (or supremum, which in fact is a maximum) of the eigenvalues of H^*H. Moreover, $\|H\| = \sqrt{\|H^*H\|}$ (see, e.g., Ref. 23) so that

$$\|H\| = \sqrt{\Lambda_s}. \tag{11}$$

If H is invertible then so is H^*H, and $\|(H^*H)^{-1}\| = {\Lambda_i}^{-1}$ where Λ_i is the greatest lower bound (or infimum) of the eigenvalues of H^*H [which is always 0 if the range of H is of infinite dimension, so that $(H^*H)^{-1}$ is unbounded]. Using $(H^{-1})^* = (H^*)^{-1}$, we can readily show that

$$\|H^{-1}\| = (\sqrt{\Lambda_i})^{-1}. \tag{12}$$

Second, the eigenvalues of H can be related to the OTF of the system. Indeed, in the discrete case the operator H is a matrix; since H is assumed, in this paper, to be a convolution operator, the matrix H has a block Toeplitz structure, which can be approximated by a block circulant matrix.[21,24] Within this approximation, which corresponds to periodizing the PSF h, H is diagonalized in the basis of the discrete Fourier exponentials $\exp[-2i\pi/N(m\mu + n\nu)]$, $0 \leqslant m,\ n \leqslant N-1$, and its eigenvalues are equal to the discrete Fourier transform values of the sampled PSF,[25] i.e., to the numerical OTF denoted hereafter by $\tilde{h}$.

Third, since the eigenvectors of H (the discrete Fourier exponentials) form an orthonormal basis, H^*H is diagonalized in the same basis as H, and its eigenvalues are the square moduli of those of H. In other words, the singular-value decomposition of H is in fact an eigenvalue decomposition, which in turn is a Fourier decomposition.

From this and Eq. (11), it is readily seen that

$$\|H\| = \max|\tilde{h}|, \quad \text{and (if } H^{-1} \text{ exists)}$$

$$\|H^{-1}\| = (\min|\tilde{h}|)^{-1}, \tag{13}$$

where $|\tilde{h}|$ is the modulation transfer function (MTF). Moreover, since the maximum value of the MTF is by convention normalized to unity (which corresponds to keeping the collecting surface constant), we obtain $\|H\| = 1$ so that the noise amplification ϵ_n and the relative noise amplification ${\epsilon_n}'$ are proportional to $c = \|G\|\|H\| = \|G\|$.

If H is invertible and if we take $G = H^{-1}$, we get

$$c = 1/\min|\tilde{h}|. \tag{14}$$

One should note that $\mathbb{H}$, the domain of definition of H, is the set of considered objects, so that the invertibility of H means that $|\tilde{h}|$ does not drop to zero on the frequency support of the considered objects. For objects with greater frequency support, H will not be invertible, and one should limit the resolution of the estimated object o_e (that is, the frequency support of the restoration filter G) to some maximum frequency $\omega_{\max}$ given *a priori* by the user (typically, the inverse of the desired resolution).

Given a maximum frequency of interest $\omega_{\max}$, we shall take for G the simple following linear filter (this filter, in operator theory terms, is the truncated singular value decomposition method[21,26]):

$$\begin{aligned} \tilde{g}(\boldsymbol{\omega}) &= 1/\tilde{h}(\boldsymbol{\omega}) \quad \text{for} \quad \boldsymbol{\omega} \in D = \{\boldsymbol{\omega}: |\boldsymbol{\omega}| \leqslant \omega_{\max}\} \\ &= 0 \quad \text{otherwise}, \end{aligned} \tag{15}$$

where $D = \{\boldsymbol{\omega}: |\boldsymbol{\omega}| \leqslant \omega_{\max}\}$ is the frequency domain of interest, here a disk of radius $\omega_{\max}$ centered at the origin. Thus parameter c is

$$c = \|G\| \times 1 = 1/\min_{\boldsymbol{\omega} \in D}|\tilde{h}(\boldsymbol{\omega})|. \tag{16}$$

Relations (8), (9), and (16) can be interpreted as follows: The noise amplification during the restoration process is (at most) proportional to c, which is the inverse of the minimum value of the MTF in the frequency domain of interest. In particular, the relative noise amplification (inverse of the SNR of the restored object) is bounded by the ratio of c over the SNR of the recorded image.

It should be noted that the actual restoration filter used when the instrument is operating will most likely be more sophisticated than this basic one, e.g., a Wiener filter. Nevertheless, it will (if it is linear) be a variation along the idea embodied by G, i.e., it will be an MTF equalizer.

The aperture configuration quality can be assessed by the value of c, and the optimization consists in finding a configuration that minimizes c, i.e., that maximizes the minimum value of the MTF over the frequency domain of interest. In this sense, the optimal configuration is the one that is the flattest, or that has the most uniform frequency coverage. Also, compact configurations arise naturally—in the present setting, where no support constraint is available—since they are the ones with finite c. And the "practical resolution limit" defined by Harvey and Rockwell[9] coincides with the maximum value of $\omega_{\max}$ for which c is finite.

It is important to note that if a support constraint is available, the relationship between the norm of an operator and the discrete Fourier transform of the corresponding kernel [as in Eq. (13)] is no longer valid. The eigenvalues of H^*H are typically "pushed upwards" by such a constraint and no longer linked to the MTF, on which zeros may then be tolerated (see Lannes[4,18] on this subject).

Finally, this optimality criterion can be refined to accommodate the possible knowledge of the statistics of the noise, as explained in Section 3.

3. REFINED CRITERION FOR KNOWN NOISE STATISTICS

If the second-order statistics of the (zero-mean) noise n are known, it is possible to derive a better estimation of the noise amplification ϵ_n than the bound given in Eq. (8). However, it should be noted that this estimate will be in expected value, whereas the bounds given in relations (8) and (9) hold for any outcome of the noise.

The restoration operator G is still assumed to be a linear filter, so that (similarly to the developments given for H) its singular values are in fact eigenvalues, which are in turn approximately equal to the discrete Fourier transform values of its sampled PSF. The singular-value decomposition of G is then a discrete Fourier decomposition, and the square of the noise amplification is given by

$${\epsilon_n}^2 = \sum_{\boldsymbol{\omega}} |\tilde{g}(\boldsymbol{\omega})|^2 |\tilde{n}(\boldsymbol{\omega})|^2, \tag{17}$$

where $\tilde{n}(\boldsymbol{\omega})$ is the Fourier transform of the noise n. Let ${\sigma_{\tilde{n}}}^2(\boldsymbol{\omega}) = E\{|\tilde{n}(\boldsymbol{\omega})|^2\}$ be the so-called average power or average intensity of $\tilde{n}(\boldsymbol{\omega})$ (Ref. 27, Sec. 9-1); taking the expected value of Eq. (17) yields

$$E(\epsilon_n{}^2) = \sum_{\boldsymbol{\omega}} |\tilde{g}(\boldsymbol{\omega})|^2 \sigma_{\tilde{n}}{}^2(\boldsymbol{\omega})$$

$$= \sum_{\boldsymbol{\omega}\in D} \frac{1}{|\tilde{h}(\boldsymbol{\omega})|^2} \sigma_{\tilde{n}}{}^2(\boldsymbol{\omega}), \tag{18}$$

where $\tilde{g}$ is the filter defined in Eq. (15). If the average power $\sigma_{\tilde{n}}{}^2(\boldsymbol{\omega})$ of the noise is known, minimizing this expression will yield an aperture configuration that is optimal in the sense that the variance of the noise amplification in the restored image will be minimal.

In particular, if n is white, then $\tilde{n}$ is stationary,[27] i.e., the average power $\sigma_{\tilde{n}}{}^2(\boldsymbol{\omega})$ is constant, so that

$$E(\epsilon_n{}^2) = \left(\sum_{\boldsymbol{\omega}\in D} \frac{1}{|\tilde{h}(\boldsymbol{\omega})|^2} \right) \times \sigma_{\tilde{n}}{}^2 \propto \left\langle \frac{1}{|\tilde{h}(\boldsymbol{\omega})|^2} \right\rangle_{\boldsymbol{\omega}\in D}, \tag{19}$$

where $\langle \cdot \rangle_{\boldsymbol{\omega}\in D}$ denotes the average on all frequencies of the support of $\tilde{g}$. The optimal configuration is then obtained by minimizing the following refined criterion c':

$$c' = \sqrt{\left\langle \frac{1}{|\tilde{h}(\boldsymbol{\omega})|^2} \right\rangle_{\boldsymbol{\omega}\in D}}. \tag{20}$$

It must be noted that if n is truly stationary, then $\tilde{n}$ is white,[27] which in particular means that $\sigma_{\tilde{n}}{}^2(\boldsymbol{\omega})$, which is equal to the autocorrelation of $\tilde{n}$ for a zero shift $E\{\tilde{n}(\boldsymbol{\omega})\tilde{n}^*(\boldsymbol{\omega} + \mathbf{0})\}$, is infinite. This mathematical difficulty and the link between the average power $\sigma_{\tilde{n}}{}^2(\boldsymbol{\omega})$ and the power spectrum of n are explained in Appendix A.

If the noise statistics are not known, then Eq. (18) can still be used to yield the following bound:

$$E(\epsilon_n{}^2) \leqslant \frac{1}{\min_{\boldsymbol{\omega}\in D} |\tilde{h}(\boldsymbol{\omega})|^2} \times \sum_{\boldsymbol{\omega}\in D} \sigma_{\tilde{n}}{}^2(\boldsymbol{\omega}), \tag{21}$$

and the optimization reduces to the minimization of the previously derived criterion c [see Eq. (16)].

The configurations obtained with this refined criterion c' are typically slightly more compact (in the usual sense of the word) than those obtained with c, as shown in the simulations of Section 4. This can be understood intuitively as follows: On the one hand, the value of c for the c-optimal configuration is, in practice, determined by the value of the MTF at the highest frequency of interest (since the MTF typically goes down when the spatial frequency increases). On the other hand, criterion c' will "tolerate" (i.e., yield a configuration with) smaller values of the MTF at the highest frequencies (which means closer telescopes), because all frequencies of interest contribute to the value of c'.

The interpretation of the values of c and c' of a given configuration is the following: c characterizes the amplification of the noise at the spatial frequency inside the domain of interest that is the most attenuated by the optical system, whatever the noise statistics may actually be, whereas c' characterizes the average amplification of the noise at all spatial frequencies within the domain of interest for a white noise.

4. SIMULATIONS

The above criteria for aperture configuration optimality were implemented and tested for a given maximum frequency of interest for three, four, and five telescopes that are assumed to be identical and whose diameter is determined by the total collecting surface, which is kept constant in all simulations. In order to express the diameter of the telescopes in the same unit as the maximum frequency, the latter is best expressed as an equivalent length, namely, the wavelength divided by the angular resolution. This length is called the maximum-frequency equivalent diameter. With this convention, the cutoff frequency D/λ of a monolithic telescope of diameter D (of, say, 40 arbitrary units) would be taken as 40. Thus, diameters and frequencies will be expressed in the same arbitrary units—pixels—in the following.

In the presented simulations, the telescopes are constrained to lie on a circle, whose radius is allowed to vary. This is especially justified in order to simplify the design of an optical space instrument[8,11,13] and also limits the search of the algorithm (except for three telescopes, since three nonaligned points are always on a circle).

Since the elementary telescopes are assumed to be identical, the global OTF of the array, OTF, can be computed as the sum of replicas of the elementary telescope OTF, OTF_e, placed at the correlation peaks of the array:

$$OTF = OTF_e \star RTF, \tag{22}$$

where RTF is the radio transfer function, a set of delta functions placed at the correlation peaks of the array, the one at the origin being of height 1, and the others of height given by the redundancy (e.g., $1/N_t$ for a nonredundant array, where N_t is the number of telescopes). This computation is more accurate than performing a numerical correlation of the aperture, especially for frequencies close to the cutoff. The elementary telescope OTF is computed by the analytical formula given in Appendix B to take into account the central obscuration.

The algorithm used is essentially an exhaustive search of the possible positions of the telescopes. The search is pruned by keeping the first of the N_t telescopes fixed and by limiting the angle between the first two telescopes to $2\pi/N_t$; indeed, if we consider a configuration in which this angle is larger than $2\pi/N_t$, there will exist another pair of telescopes separated by an angle smaller than this

Table 1. Optimal Configurations with Criterion c[a]

Number of Telescopes	c	c'	Support Diameter (pixels)	Angular Position (deg)
3	26.1	8.9	72	−180, −60, 60
4	13.8	7.7	72	−180, −120, −32, 92
5	13.6	7.0	74	−180, −108, −37, 35, 108

[a] The second column (c) gives a bound for noise amplification at any frequency; the third (c') gives the average noise amplification at all frequencies. The fourth and fifth columns give the diameter of the circle supporting the telescopes and their angular positions, respectively. The maximum frequency of interest is 70 pixels; the collecting surface is kept constant and corresponds to a telescope diameter of 40 pixels for three telescopes; all telescopes have a 33% central obscuration.

Table 2. Optimal Configurations with Criterion c'[a]

Number of telescopes	c	c'	Support Diameter (pixels)	Angular Position (deg)
3	31.1	7.5	62	−180, −59, 59
4	21.1	7.1	64	−180, −92, −26, 64
5	14.7	6.7	66	−180, −108, −35, 35, 108

[a]The second column (c) gives a bound for noise amplification at any frequency; the third (c') gives the average noise amplification at all frequencies. The fourth and fifth columns give the diameter of the circle supporting the telescopes and their angular positions, respectively. The maximum frequency of interest is 70 pixels; the collecting surface is kept constant and corresponds to a telescope diameter of 40 pixels for three telescopes; all telescopes have a 33% central obscuration.

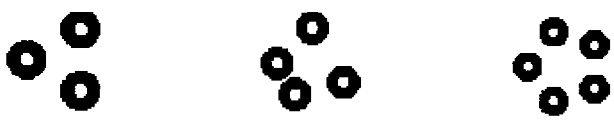

Fig. 1. Optimal configurations with criterion c for three, four, and five telescopes (see Table 1).

Fig. 2. Optimal configurations with criterion c' for three, four, and five telescopes (see Table 2).

value, so that the configuration will be equivalent, apart from a rotation, to one in which the first two telescopes are less than $2\pi/N_t$ away.

In all our simulations the scale was set by taking 20 pixels as the radius of the telescopes of the three-telescope array. The radii of the telescopes of the other arrays are determined by keeping the collecting surface constant. We considered a typical value of 0.33 for the central obscuration and a frequency of interest of 70. The angular increment is 2 deg, and the radial increment is at most 2 pixels. The optimal configurations for three, four, and five telescopes are shown in Tables 1 (criterion c) and 2 (criterion c'), and are depicted in Figs. 1 and 2, respectively.

For this value of $\omega_{\max}$ and criterion c (Table 1), the five-telescope optimal configuration is slightly better than the four-telescope configuration, which in turn is far better than the three-telescope configuration.

For criterion c' also (Table 2), the five-telescope configuration is better than the four-telescope one, which in turn is better than the three-telescope one, but the differences in the values of c' are much smaller, owing to the average in the computation of c'.

For three telescopes the optimum configuration is an equilateral triangle, but for four telescopes it is not a square. For five telescopes the optimum configuration is a regular pentagon (within the precision of our simulations). In other words, the chosen criterion leads to configurations that appear to minimize naturally the redundancy of the array without imposing it *a priori* in the search. One should nevertheless bear in mind that for an optical array the elementary telescopes themselves introduce some redundancy, owing to their nonzero diameter.

In order to check whether the optimal configuration we obtained (for a given number of telescopes and a given criterion) was pathological, we compared it with the ten configurations that had the closest values of the criterion (e.g., from 7.06 for the optimum to 7.09 for the tenth-best four-telescope configuration according to criterion c'). We indeed noticed that all these configurations were very close to the one that ranked first, apart from a possible symmetry and/or rotation.

Even for the simple case of three telescopes, criterion c or c' can be useful in determining the optimal spacing between telescopes and the expected quality of the frequency coverage for a given resolution and telescope diameter. For three telescopes the optimum configuration is always an equilateral triangle. Their spacing increases with the desired resolution (or maximum fre-

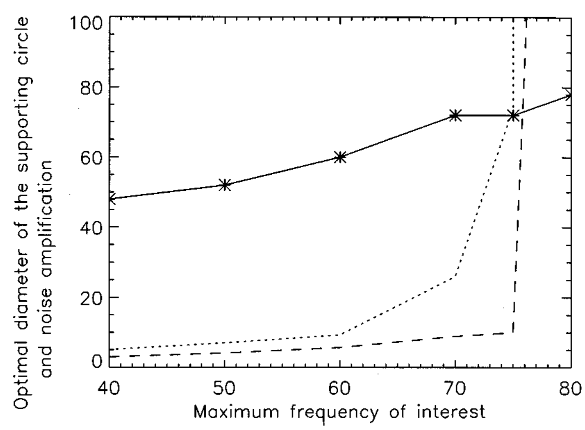

Fig. 3. Evolution of the optimal spacing of a three-telescope array with the maximum frequency (solid curve) and corresponding noise amplification parameters c (dotted curve) and c' (dashed curve). The optimization is done with criterion c, the telescope diameter is 40 pixels, and the frequency is expressed in pixels.

Table 3. Optimal Diameter of the Circle Supporting the Telescopes and Noise Amplification Parameters as a Function of the Maximum Frequency, for a Three-Telescope Array[a]

Maximum Frequency (pixels)	Noise Amplification Parameters		Optimal Diameter of the Support (pixels)
	c	c'	
40	5.1	3.0	48
50	7.0	4.1	52
60	9.3	5.7	60
70	26.0	8.9	72
75	74.0	10.0	72
80	4×10^5	4×10^2	78
90	∞	∞	NA

[a]Telecopes are of diameter 40. Optimization is done with criterion c. NA, not applicable.

quency), as shown in Fig. 3. As expected intuitively, since the telescopes are constrained to lie on a circle, the optimal diameter of the circle supporting the telescopes is never far from the diameter of a monolithic telescope having the maximum frequency of interest as its cutoff frequency. Also, not surprisingly, the quality of the frequency coverage degrades when the maximum frequency increases, and this quality can be quantified by parameters c and c' (see Fig. 3 and summary Table 3). Finally, if the maximum frequency of interest is too high (in practice above 80 in the simulations presented here), no compact configuration can reach it, so that c and c' become infinite.

5. CONCLUSION

A criterion has been derived to find an optimal aperture configuration for a synthetic-aperture optical instrument that provides images of extended objects. This criterion is based on the minimization of the restoration error: the difference between the original object and the one that will be estimated from the recorded image. It explicitly takes into account the resolution to be achieved and the diameter of the elementary telescopes. This criterion was shown to be equal to the inverse of the minimum value of the MTF in the frequency domain of interest. It gives a bound on the noise amplification in the imaging-plus-restoration process at all frequencies of interest, for any outcome of the noise. Furthermore, it has been refined to take into account the possible knowledge of the noise second-order statistics. For a white noise the refined criterion is the average of the inverse of the squared MTF in the frequency domain of interest. It is proportional to the expected value of the noise amplification.

The results of the optimization of arrays of three, four, and five telescopes, under the constraints of a given collecting surface and resolution, can be summarized as follows: First, the three- and five- telescope optimal configurations are regular polygons, whereas the four-telescope one is not. We interpret this as being because our criterion naturally minimizes the redundancy of the array without imposing it *a priori*. Second, the optimal diameter of the circle supporting the telescopes was found always to be close to that of a monolithic telescope having the maximum frequency of interest as its cutoff.

Future work should assume an optimal linear (i.e., Wiener) filter instead of the truncated inverse filter used here, even if the truncated inverse filter is close to the Wiener filter in the case of a high SNR. Additionally, in our approach, the SNR is controlled through the choice of the collecting surface. An extension of this work could then include the incorporation of constraints, such as positivity and/or support (when the object's support lies within the field of view), to allow zeros in the transfer function while still taking into account the frequency coverage of the elementary telescopes.

APPENDIX A: FOURIER TRANSFORM OF A STATIONARY NOISE

In this appendix we examine the relationship between the power spectrum of the noise n and the average power of its Fourier transform $\tilde{n}$. See Papoulis (Ref. 27, Sec. 11-3) for an overview of the second-order properties of the Fourier transform of random processes.

Let n be a noise (a zero-mean stochastic process, of the one-dimensional variable t for the sake of clarity). The Fourier transform $\tilde{n}(\omega)$ of $n(t)$ is also a zero-mean stochastic process of the variable ω. The autocorrelation of $\tilde{n}(\omega)$ is $E\{\tilde{n}(\omega_1)\tilde{n}^*(\omega_2)\}$, and, in particular, the average power or average intensity of $\tilde{n}(\omega)$ is by definition (Ref. 27, Sec. 9-1) ${\sigma_{\tilde{n}}}^2(\omega) = E\{|\tilde{n}(\omega)|^2\}$.

If $n(t)$ is white, then $\tilde{n}(\omega)$ is stationary, i.e., ${\sigma_{\tilde{n}}}^2(\omega)$ does not depend on ω. Conversely, if n is stationary, then $\tilde{n}$ is white; i.e., its autocorrelation is a Dirac function, and in particular the average power ${\sigma_{\tilde{n}}}^2$ is infinite. The origin of this infinite value is that a stationary noise is implicitly observed from $t = -\infty$ to $t = +\infty$.

Indeed, let n_T be the observed noise, equal to the true noise n (assumed to be stationary) in the observation time window $[-T/2, T/2]$ and zero outside this window. Then n_T is not stationary and its average power is finite, and we can express it as a function of the power spectrum of the noise n. Let $\Pi_T(t)$ be the characteristic function of the observation time window and $R(\tau)$ be the autocorrelation of n; then the autocorrelation of n_T and $R(\tau)$ are related through

$$E[n_T(t)n_T(t + \tau)] = \Pi_T(t)\Pi_T(t + \tau)R(\tau). \quad \text{(A1)}$$

If the support of R is much smaller than the observation time window, then

$$E[n_T(t)n_T(t + \tau)] \approx \Pi_T(t)R(\tau). \quad \text{(A2)}$$

The autocorrelation of the Fourier transform of the observed noise is then

$$\begin{aligned} E[\tilde{n}_T(\omega_1)\tilde{n}_T^*(\omega_2)] &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} E\{n_T(t)n_T(t + \tau)\}\exp[-2i\pi(\omega_1-\omega_2)t]\exp[2i\pi\omega_2\tau]\mathrm{d}t\,\mathrm{d}\tau \\ &\approx \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \Pi_T(t)\exp[-2i\pi(\omega_1-\omega_2)t]R(\tau)\exp(2i\pi\omega_2\tau)\mathrm{d}t\,\mathrm{d}\tau \\ &\approx T\ \mathrm{sinc}\{(\omega_1 - \omega_2)T\}S(\omega_2), \end{aligned} \quad \text{(A3)}$$

where $S(\omega)$ is the power spectrum of the true noise n, i.e., the Fourier transform of $R(\tau)$. This expression, as expected, shows that the autocorrelation of $\tilde{n}_T$ tends towards a Dirac function when the observing time tends to infinity. It shows in particular that the average power of

$\tilde{n}_T$ is finite and directly proportional to the power spectrum of the noise (a result derived in a different manner in Ref. 27):

$$\sigma_{\tilde{n}_T}{}^2(\omega) = TS(\omega). \tag{A4}$$

Of course, if $n(t)$ is additionally white, then $\sigma_{\tilde{n}_T}{}^2(\omega)$ is constant.

APPENDIX B: OPTICAL TRANSFER FUNCTION WITH CENTRAL OBSCURATION

To the best of our knowledge, an analytic expression of the OTF of a telescope with central obscuration was first derived by Perrier.[28] Unfortunately, the expression appeared with what is most likely a typographic error (in the expression for function H_1). Thus we take this opportunity to derive an equivalent and somewhat simpler expression.

The basis of the computation is to derive the correlation $C_U(x)$ between two disks, one of diameter 1 and one of diameter U, where U is the (linear) central obscuration, defined as the ratio of the obscuration diameter to the telescope pupil diameter. Geometric considerations and elementary trigonometry yield the following result [apart from a $4/\pi$ normalization factor to ensure that $C_1(0) = 1$]:

$$C_U(x) = \begin{cases} U^2 & : x \leqslant \dfrac{1-U}{2} \\[2ex] 0 & : x \geqslant \dfrac{1+U}{2} \\[2ex] \dfrac{1}{\pi}\arccos\left(x + \dfrac{1-U^2}{4x}\right) & \\ \quad + \dfrac{U^2}{\pi}\arccos\left[\dfrac{1}{U}\left(x - \dfrac{1-U^2}{4x}\right)\right] & : \dfrac{1-U}{2} \leqslant x \leqslant \dfrac{1+U}{2} \\ \quad - \dfrac{2x}{\pi}\sqrt{1 - \left(x + \dfrac{1-U^2}{4x}\right)^2} & \end{cases}. \tag{B1}$$

In addition, the pupil function of a telescope with central obscuration can be written as $P = P_+ - P_-$, where P_+ is the pupil function of the telescope without central obscuration and P_- is the pupil function of the central obscuration (1 inside the obscuration, and 0 outside). The OTF of the telescope with central obscuration reads as

$$OTF = P \otimes P = P_+ \otimes P_+ + P_- \otimes P_- - 2P_+ \otimes P_-, \tag{B2}$$

where $\otimes$ denotes correlation. With proper normalization the OTF is then readily expressed as a function of C_U:

$$OTF(x) = \frac{1}{1-U^2}[C_1(x) + U^2 C_1(x/U) - 2C_U(x)], \tag{B3}$$

where $x = \lambda f/D$ is the reduced frequency.

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