Efficient phase estimation for large-field-of-view adaptive optics

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We propose a maximum *a posteriori*–based estimation of the turbulent phase in a large field of view (FOV) to overcome the anisoplanatism limitation in adaptive optics. We show that, whatever the true atmospheric profile, a small number of equivalent layers (two or three) is required for accurate restoration of the phase in the whole FOV. The implications for multiconjugate adaptive optics are discussed in terms of the number and conjugated heights of the deformable mirrors. The number of guide stars required for wave-front measurements in the field is also discussed: three (or even two) guide stars are sufficient to produce good performance. © 1999 Optical Society of America

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Anisoplanatism is one of the most severe limitations on phase correction by adaptive optics after propagation through the turbulent atmosphere. Obtaining high-resolution images in a large field of view (FOV) requires new approaches for a good estimation of the phase in the whole field. Tallon and Foy[1] and Ragazzoni *et al.*[2] suggested a tomographic approach, which consists of the reconstruction of the whole turbulence volume by use of several natural or laser guide stars (GS) for wave-front sensing. Based on the same idea, a multiconjugated adaptive optics (MCAO) system was studied by Beckers,[3] Ellerbroek,[4] and Johnston and Welsh,[5] who used several conjugated mirrors to compensate for the turbulence at different heights. For practical reasons it is impossible to consider a large number of mirrors; thus we are led to the critical question: "Can we model the phase variation in the field by using an atmospheric model based on a very small number of thin layers?" The answer to this question is the key issue addressed in this Letter. We consider here, for simplicity, natural GS's, but the case of laser GS's is also discussed.

We present a method to estimate the phase in a large FOV through a model that incorporates a small number of turbulent layers. This method is validated in simulation. The implications for a MCAO system are then discussed.

Let us consider a true C_n^2 profile sampled by N_{true} thin layers. In practice, the C_n^2 profile can be measured by scintillation detection and ranging before the observations are made.[6] In the so-called small-perturbation approximation, the phase $\Phi_{\boldsymbol{\alpha}}(\mathbf{r})$ on the telescope pupil in angular direction $\boldsymbol{\alpha}$ is given by

$$\Phi_{\boldsymbol{\alpha}}(\mathbf{r}) = \sum_{j=1}^{N_{\mathrm{true}}} \varphi_j(\mathbf{r} + h_j \boldsymbol{\alpha}), \tag{1}$$

where φ_j and h_j are, respectively, the phase screen and the height of the jth layer and $\mathbf{r}$ is the position vector in the telescope pupil. For a radius of the FOV of $\alpha_{\max}$, the support of φ_j is a disk of diameter $D_j = D + 2h_j\alpha_{\max}$ in a given layer j, where D is the telescope diameter. Let $\boldsymbol{\rho}_j$ be the position vector in this disk.

A crude turbulent distribution model by only N_{EL} equivalent layers (EL's) (with $N_{\mathrm{EL}} \le N_{\mathrm{true}}$) is proposed. The true C_n^2 is divided into N_{EL} slabs. For the first slab, of 2-km thickness for the simulations in this paper, an EL is placed on the telescope pupil to model the turbulence near the ground. The other slabs are regularly spaced. For each slab, lying between $h_{j,\min}$ and $h_{j,\max}$, an EL is placed at an equivalent height $h_{\mathrm{eq},j}$, defined as the weighted mean height of the jth slab: $h_{\mathrm{eq},j} = [\int_{h_{j,\min}}^{h_{j,\max}} C_n^2(h) h \mathrm{d}h]/[\int_{h_{j,\min}}^{h_{j,\max}} C_n^2(h)\mathrm{d}h]$, with an associated $r_{0,j} \propto [\int_{h_{j,\min}}^{h_{j,\max}} C_n^2(h)\mathrm{d}h]^{-3/5}$.

The idea is to use this N_{EL} EL model in a maximum *a posteriori*– (MAP-) based approach to estimate the phase in a large FOV. The goal of this approach is to find the unknowns, i.e., the most likely N_{EL} phase screens $\varphi_j(\boldsymbol{\rho}_j)$, given the data $\{\Phi^m_{\boldsymbol{\alpha}_i}(\mathbf{r})\}_i$, i.e., the pupil phase maps $\Phi^m_{\boldsymbol{\alpha}_i}(\mathbf{r})$ measured for a discrete set of GS directions $\boldsymbol{\alpha}_i$. Applying Bayes's rule shows that the so-called *a posteriori* probability is proportional to the product of the likelihood of the data and the *a priori* probability of the unknowns. Therefore the probability law that must be maximized with respect to $\{\varphi_j(\boldsymbol{\rho}_j)\}_j$ (with $j \in [0, N_{\mathrm{EL}}]$) reads as

$$P[\{\varphi_j(\boldsymbol{\rho}_j)\}_j | \{\Phi^m_{\boldsymbol{\alpha}_i}(\mathbf{r})\}_i] \propto P[\{\Phi^m_{\boldsymbol{\alpha}_i}(\mathbf{r})\}_j | \{\varphi_j(\boldsymbol{\rho}_j)\}_j] P[\{\varphi_j(\boldsymbol{\rho}_j)\}_j].$$

The likelihood term, which accounts for the noise that affects the wave-front measurements, can be rewritten as

$$P[\{\Phi^m_{\boldsymbol{\alpha}_i}(\mathbf{r})\}_i | \{\varphi_j(\mathbf{r})\}_j] \propto \prod_{i=1}^{N_{\mathrm{GS}}} \exp\{-1/2 [\Psi_{\boldsymbol{\alpha}_i}(\mathbf{r})]^T C_i^{-1} [\Psi_{\boldsymbol{\alpha}_i}(\mathbf{r})]\}, \tag{2}$$

where $\Psi_{\boldsymbol{\alpha}_i}(\mathbf{r}) = [\Phi^m_{\boldsymbol{\alpha}_i}(\mathbf{r}) - \sum_{j=1}^{N_{\mathrm{EL}}} \varphi_j(\mathbf{r} + h_j \boldsymbol{\alpha}_i)]$ and N_{GS} is the number of GS's, i.e., of wave-front measurements. C_i^{-1} is the covariance matrix of the noise for the GS

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i, which is assumed to be Gaussian and decorrelated between measurements.

The *a priori* term includes the prior knowledge of the phase statistics; assuming Gaussian statistics, it is given by

$$P[\{\varphi_j(\boldsymbol{\rho}_j)\}_j] = \prod_{j=1}^{N_{\mathrm{EL}}} \exp[-{}^1\!/_2 \varphi_j{}^T(\boldsymbol{\rho}_j) C_{\mathrm{Kol},j}{}^{-1} \varphi_j(\boldsymbol{\rho}_j)], \tag{3}$$

where the N_{EL} phase screens are assumed to be statistically independent. Each phase screen follows Kolmogorov statistics; hence the covariance matrix $C_{\mathrm{Kol},j}$ scales according to $r_{0,j}$. Finally, the phase is estimated by the minimization of

$$J[\{\varphi_j(\boldsymbol{\rho}_j)\}_j] = \sum_{i=1}^{N_{\mathrm{GS}}} [\Psi_{\boldsymbol{\alpha}_i}(\mathbf{r})]^T C_i{}^{-1} [\Psi_{\boldsymbol{\alpha}_i}(\mathbf{r})] + \sum_{j=1}^{N_{\mathrm{EL}}} \varphi_j(\boldsymbol{\rho}_j)^T C_{\mathrm{Kol},j}{}^{-1} \varphi_j(\boldsymbol{\rho}_j) \tag{4}$$

with respect to the phase screens φ_j. Note that the minimization of such a criterion is quadratic and thus leads to an analytical solution.

To study the influence of the number of EL's to be considered, i.e., the influence of the sampling step, requires a simulation with different atmospheric profiles. Here we present the results obtained with a profile inspired from measurements made at Mauna Kea (Hawaii) by Racine and Ellerbroek[7] (see Fig. 1). The true profile here comprises $N_{\mathrm{true}} = 16$ layers. We studied other profiles, including a constant $C_n{}^2$ between 0 and 15 km, to test the robustness of the method. The results are similar to the ones presented here.

The phase screens on each layer are simulated by Roddier's method[8] by use of the first 300 Zernike polynomials (radial order up to 23). The size of these phase screens corresponds to a 56-arcsec FOV ($\alpha_{\max} =$ 28 arcsec) and a telescope diameter of 4 m. The overall D/r_0 is 6 (typical result in the K band for 0.92-arcsec seeing). The phase measurements are made at five field positions $\{\boldsymbol{\alpha}_i\}$ that are located at the five vertices of a regular pentagon inscribed in a circle of radius $\alpha_{\max}$. The GS's are assumed to be natural GS's; that is, the laser GS–specific problems, cone effect and tilt estimation, are not taken into account. This rather favorable GS configuration[1] allows us first to study the phase error that is due solely to undersampling of the turbulence profile. The measurements are the true phases plus a noise. The noise level corresponds to a 7×7 subaperture Shack–Hartmann sensor with a signal-to-noise ratio (SNR) equal to 1.8 (ratio between the turbulent variance of the angle of arrival in a subaperture to the noise variance). The noise variance on the Zernike coefficients evolves[9] as (radial order)$^{-2}$, and, for the SNR considered, it becomes greater than the Kolmogorov turbulent variance[10] [which evolves as (radial order)$^{-11/3}$] after the 21st Zernike polynomial. For the restoration in each layer, we denote by $\hat{\varphi}_j$ the estimated phase, which is expanded on the first 300 Zernike polynomials. Equation (4) is easily transposed on this basis. The use of Zernike coefficients allows us to incorporate the Kolmogorov statistics[11] on each EL.

The MAP-based restoration is applied for different numbers of EL's. The performance of the method is evaluated in terms of a Strehl ratio (SR) of $\exp[-\sigma_{\mathrm{err}}{}^2(\boldsymbol{\alpha})]$, where $\sigma_{\mathrm{err}}{}^2(\boldsymbol{\alpha})$ is computed by

$$\sigma_{\mathrm{err}}{}^2(\boldsymbol{\alpha}) = \left\langle \frac{1}{S} \int_S \left[\Phi(\mathbf{r}, \boldsymbol{\alpha}) - \sum_{j=1}^{N_{\mathrm{EL}}} \hat{\varphi}_j(\mathbf{r} + h_j \boldsymbol{\alpha}) \right]^2 \mathrm{d}\mathbf{r} \right\rangle, \tag{5}$$

where S is the pupil surface. In Fig. 2 we present the SR variation as a function of α. Considering the particular GS geometry, we have chosen a cut of the field including the worst and the best SR's in the field.

Figure 2 shows the good reconstruction of the phase in the whole FOV when five GS's and our approach are used. The curves for three, four, and five EL's are indistinguishable. The SR is high and nearly constant. For comparison, we show a conventional MAP estimation with a single on-axis GS ($\boldsymbol{\alpha} = 0$) and one EL on the telescope pupil phase optimized for on-axis observation and applied in the whole FOV. In this case the SR decreases rapidly as a function of angle beyond 10 arcsec because of the effects of anisoplanatism. Note that, in this conventional case,

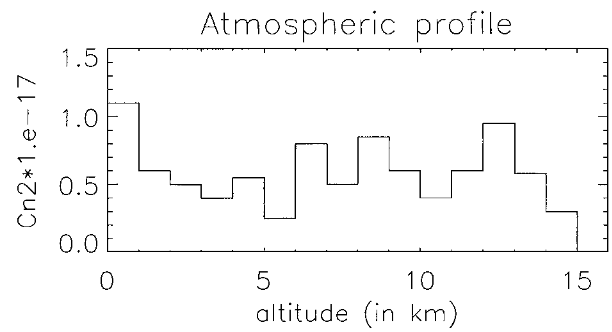

Fig. 1. $C_n{}^2$ true profile used in the simulation; $C_n{}^2 = 0$ above 15 km. Telescope altitude, 0 km.

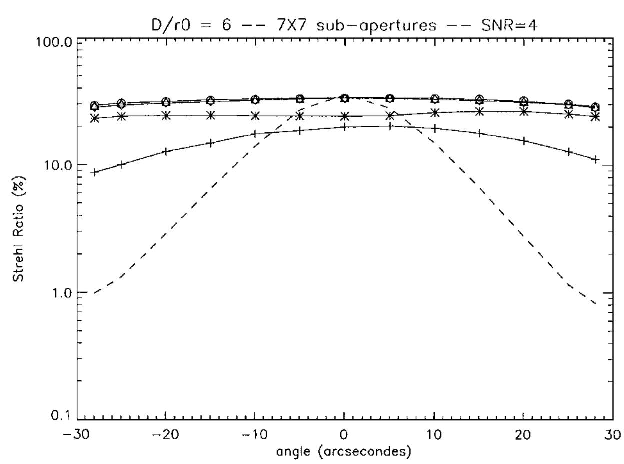

Fig. 2. Influence of the number of EL's on SR(α) (in percent). Dashed curve, conventional on-axis single GS and one EL on the telescope pupil. With 5 GS's: (+) one EL at 6.5 km; (*) two EL's (0 and 8.5 km); (◇) three EL's (0, 5.4, and 11.7 km); (△) four EL's (0, 4.2, 8.4, and 12.5 km); (○) five EL's (0, 3, 6.8, 10.7, and 13.7 km).

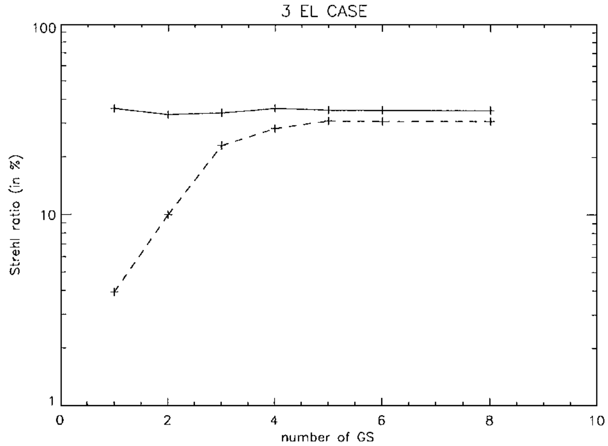

Fig. 3. SR (in percent) versus number of GS's: the maximum (solid curve) and the minimum (dashed curve) values in the FOV. Three EL's are considered here.

we have chosen a SNR of 4 on the Shack–Hartmann sensor to ensure having the same maximum SR as in the optimal EL case.

The other important point is that an increase in the number of EL's (N_{EL}) does not significantly improve the phase restitution as soon as $N_{EL} \geq 3$, even if, when $N_{EL} \rightarrow N_{true}$, this approach tends to the true MAP and leads to the tomographic scheme.[1,2] The key point to be emphasized is that only a few EL's (two or three), i.e., a loose sampling of the turbulence profile, are necessary for precise modeling of the phase in a large FOV. Of course, these results depend on the ratio FOV $\times\, h_{max} \times 1/\cos z$ (here h_{max} is 15 km and z is the zenith angle) to the pupil diameter. When this ratio decreases, the gain achieved by the use of three EL's is reduced to that obtained with two EL's.

Such a result is particularly important for MCAO systems.[4,5,7] We have demonstrated that even with rather uniform $C_n{}^2$ profiles it is not necessary to estimate (and thus to correct) the phase in each turbulent layer of the profile but to do so only in a very small number of EL's. The number of required conjugate mirrors in a MCAO system is therefore quite small; e.g., two mirrors already permit good correction and three provide a quasi-optimal correction. Note that one mirror conjugated at 6.5 km (Ref. 7) and five GS's already lead to a substantial gain in the whole FOV even if the SR on axis is lower than in the conventional case.

Another scaling parameter to study for the design of a large-FOV high-resolution system is the number of GS's that are necessary to produce a good estimation of the phase for a given FOV. We apply our approach with three EL's and a variable number of GS's, one on axis, two at ±28 arcsec, and three–eight at the vertices of regular polygons inscribed in the FOV (radius, 28 arcsec). For each configuration we estimate the maximum and the minimum SR in the whole field. Again, the SNR for each GS configuration is therefore chosen to yield the same maximum SR. The results are presented in Fig. 3. As soon as the number of GS's is greater than two, the phase estimation quality is nearly uniform in the whole field, and an increase in this number of GS's does not significantly improve the performance. Tallon and Foy[1] proposed using four GS's, but the present study shows that three (or even two for an elongated FOV) may be enough, depending on the SR requirements in the field. Note that an array of laser GS's may be used for the wave-front measurements if no natural GS is available.[1,2,4,5] In that case the conical effect and the tilt estimation problem must be addressed. These specific limitations might degrade the performance, but the reconstruction principle is still valid, and the conclusion concerning the number of required mirrors should be unchanged.

This study allows us to define the characteristics and the expected performance of large-FOV high-resolution imaging systems. We have shown that, whatever the true $C_n{}^2$ profile, three EL's provide a quasi-optimal restoration of the phase in the whole FOV and that even only two layers are enough to produce a good and nearly uniform reconstruction. Therefore full tomography of the atmosphere is not necessary. In addition, only three (or even two) GS's are required for such an imaging system. Because of the weak dependency of the angular decorrelation of the phase on the atmospheric profile, the positions of the EL's are not critical; i.e., low precision is required on the $C_n{}^2$ profile. A change of a few kilometers in the EL positions leads to a SR variation of only the order of 1%, with the same noise level as above. The MAP-based approach presented here can be applied directly for image postprocessing (deconvolution from wave-front sensing,[12] phase diversity[13]), and the results can be generalized to MCAO systems. We are currently studying the optimal number of actuators for each conjugated deformable mirrors and *a priori* for the closed-loop phase statistics.

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