# A NEW METHOD FOR ADAPTIVE OPTICS POINT SPREAD FUNCTION RECONSTRUCTION

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**Abstract.** Images obtained with adaptive optics (AO) systems can be improved by using restoration techniques ; the OA correction being only partial. However, these methods require an accurate knowledge of the system point spread function (PSF). Telemetry data of adaptive optics systems can be used to estimate the PSF during the science observation. [Veran et al.(1997)] developed a method to characterize statistically the residual phase of the AO system and to estimate the PSF using data acquired by the system during the observation. In this paper, we present a new method to estimate the AO PSF based on a maximum likelihood approach. This method can be used to estimate simultaneously the covariance of the residual phase including the propagation of the noise on the deformable mirror (DM) and the Fried parameter $r_0$ during the observation. We present in the paper the theory of the method, an approach to get an estimate of the different prior to feed the method, and the first results obtained on simulated data of a Canary-like system [Gendron et al.(2011)] in closed-loop mode.

## 1 Introduction

To get the best image quality from an optical system in terms of contrast, it is required to deconvolve the image from the point spread function (PSF). For this method to be effective, it is necessary that the optical transfer function (OTF) of the system telescope plus atmospheric residuals is well characterized. The optimal situation is to have a nearby star in the isoplanatic field around the science target which provides a PSF that has undergone the same correction under the same conditions of turbulence. The OTF can also be obtained by observing separately, but close in time and space, a star with similar characteristics than the science target to get an estimate of the PSF consistent with the observation. Unfortunately, this situation requires a large amount of integration time for the PSF calibration.

PSF reconstruction is another solution. The AO loop measures continuously the state of the turbulence. It is then possible to use a method to reconstruct the OTF (or the PSF) of the residual atmosphere during the observation using data from real-time telemetry. The reconstruction is done in post-processing and does not require extra time.

[Veran et al.(1997)] developed the first method for PSF reconstruction and applied successfully the method to the PUEO system at CFHT [Arsenault et al.(1994)]. This method is based on a least-squares (LS) approach and requires an estimate of the noise variance and an estimate of the covariance of the aliasing on the mirror modes so as to unbias the covariance matrix of the modes. But it is based on a number of approximations limiting its accuracy.

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## 2 Estimation of the covariance of the modes: a new maximum likelihood approach

We use a probabilistic approach to estimate $C_\epsilon$ the covariance matrix of the modes of the residual phase $C_\epsilon$ in a basis containing an infinite number of modes to explicitly take into account the effects of aliasing on the measurement, of noise and of the loop temporal bandwidth. The method of maximum likelihood (ML) is applicable in our case [Gratadour (2005)]. This approach provides good results when trying to estimate a small number of parameters for a large number of data.

Let us consider a linear model of the measurement **w**:

$$w = \mathbf{D}\epsilon + n_w. \tag{1}$$

The noise $n_w$ on the measurement is white Gaussian. Since the residual phase $\epsilon$ is composed of many random processes (central limit theorem), passed through a linear filter, it is also Gaussian.

Thus, the probability density function of the measurements follows the multivariate normal distribution (N dimensions) defined as:

$$f(w) = \frac{1}{(2\pi)^{\frac{N}{2}} [\det(C_w)]^{\frac{1}{2}}} \exp[-\frac{1}{2}w^{\mathrm{t}}C_w^{-1}w], \tag{2}$$

with $C_w$ the covariance matrix of the measurements (Eq. 1) given by:

$$C_w =< ww^{\mathrm{t}} >= DC_\epsilon D^{\mathrm{t}} + C_n. \tag{3}$$

$C_n$ represents the covariance matrix of the noise. Because the noise is uncorrelated between each sub-aperture, $C_n = \sigma_n^2 I$, and $I$ the identity matrix and $C_\epsilon$ the covariance matrix of the residual phase $\epsilon$.

The likelihood function of the measurement if then defined as:

$$\mathcal{L}(w|C_\epsilon, \sigma_n^2) = \frac{1}{(2\pi)^{\frac{N}{2}} [\det(C_w)]^{\frac{1}{2}}} \exp[-\frac{1}{2}w^{\mathrm{t}}C_w^{-1}w], \tag{4}$$

with $N$ the number of measurements. Assuming the statistical independence between the measurement vectors at each iteration $k$, the likelihood function of the measurements time series is given by:

$$\mathcal{L}(w_i|C_\epsilon, \sigma_n^2) = \frac{1}{(2\pi)^{\frac{kN}{2}} [\det(C_w)]^{\frac{k}{2}}} \prod_{i=1}^{k} \exp[-\frac{1}{2}w_i^{\mathrm{t}}C_w^{-1}w_i], \tag{5}$$

We will maximize the likelihood function with respect to the parameters $C_\epsilon$ and $\sigma_n^2$, that is minimizing (using a conjugated gradient algorithm for example):

$$\mathcal{J}(C_\epsilon, \sigma_n^2) = -\ln(\mathcal{L}) \propto \frac{k}{2}\ln(\det[C_w]) + \frac{1}{2}\sum_{i=1}^{k} w_i^{\mathrm{t}}C_w^{-1}w_i$$

$$\propto \frac{k}{2}\ln(\det[DC_\epsilon D^{\mathrm{t}} + C_n]) + \frac{1}{2}\sum_{i=1}^{k} w_i^{\mathrm{t}}(DC_\epsilon D^{\mathrm{t}} + C_n)^{-1}w_i. \tag{6}$$

# 3 The covariance matrix $C_\epsilon$

In this work, we used the Karhunen-loéve (KL) modes as a basis for the expansion of the phase. Obviously, an infinite number of modes is impossible to compute and we thus use 200 of these modes. At each iteration during the simulation, the phase is projected (with a least-squares projection) on the modes such that:

$$\Phi_\epsilon = \epsilon KL, \tag{7}$$

$$KL^+ = (KL^t KL)^{-1} KL^t, \tag{8}$$

where $KL$ is the influence matrix. Thus, we have:

$$KL^+ \Phi_\epsilon = \epsilon \tag{9}$$

We chose to simulate an AO system compensating for the 30 first KL from the 200 KL in the basis. These 30 modes are hereafter called the mirror modes.
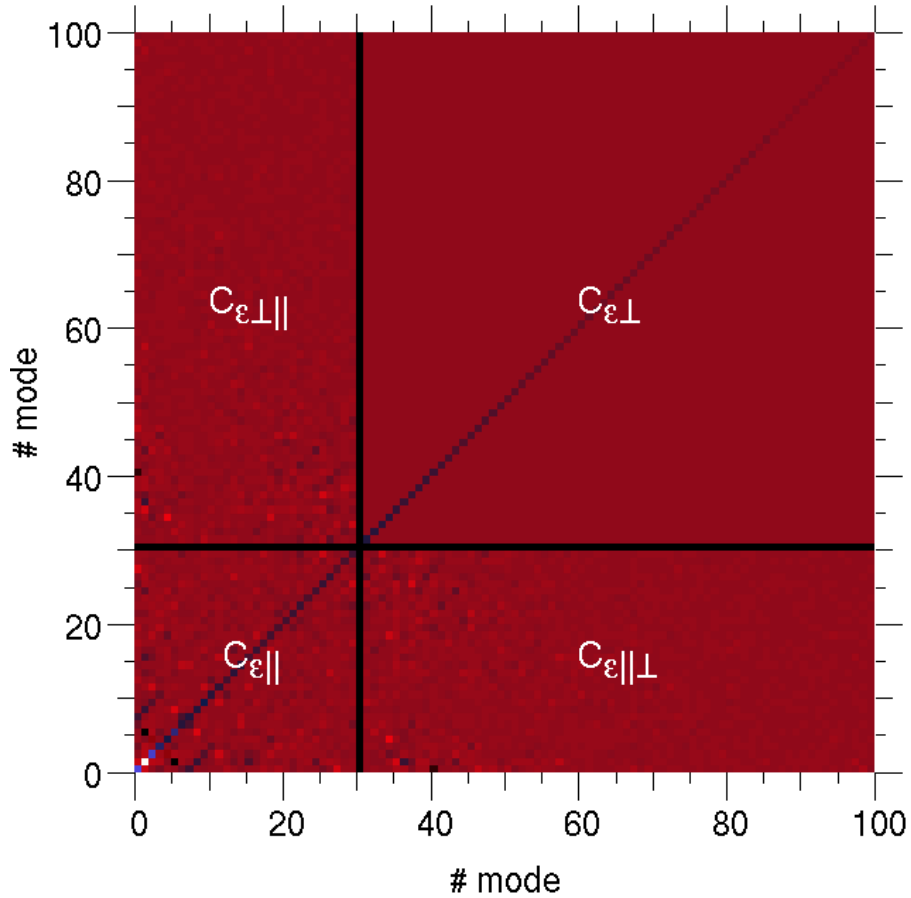


**Fig. 1.** Covariance matrix of the modes for 100 KL. The complete matrix contains 200 KL.

The vector $\epsilon$ is the expansion vector of the residual phase ($\Phi_\epsilon$) in the 200-KL basis. The covariance matrix is then defined as:

$$C_\epsilon = < \epsilon \epsilon^t > . \tag{10}$$

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This matrix contains $200 \times 200$ elements and is shown Fig. 1. We decompose the matrix in 4 blocks, limited by the number of DM modes (i.e. 30 in our simulation):

– $C_{\epsilon\perp}$: This term represents the correlations between high order modes uncorrected by the mirror. KL modes being statistically independent, the covariance matrix is diagonal. Assuming a Kolmogorov phase, the covariance of these modes depends only on the parameter $r_0$ and can be calculated once and for all by simulation assuming (for instance) a unitary $D/r_0$ ratio with $D$ is the telescope diameter. We can then deduce the covariance for a particular observation:

$$C_{\epsilon\perp}\Big|_{\frac{D}{r_0}} = C_{\epsilon\perp}\Big|_{\frac{D}{r_0}=1} \times \left(\frac{D}{r_0}\right)^{5/3}. \tag{11}$$

– $C_{\epsilon\|}$: because of the correction of the AO system, correlations between the 30 first modes appear: the covariance matrix of the modes is not diagonal. These terms are difficult to estimate. They stand for the residual component in the mirror space including the temporal residuals, the aliasing and the propagation of measurement noise. In addition, they depend on the correction performance.

– $C_{\epsilon\|\perp}$: the cross term is the coupling between the modes of high spatial frequencies and modes in the mirror space (corrected). Out of AO correction, these terms should be zero because the orthogonal phase and the parallel phase are statistically independent. However, due to the aliasing on the sensor and the finite temporal bandwidth, erroneous commands are sent to the DM. To get the covariance $C_{\epsilon\|}$ from the covariance of measurements, it is necessary to have an estimate of the terms $C_{\epsilon\|\perp}$.

– $C_{\epsilon\perp\|}$ is the transposed matrix of $C_{\epsilon\|\perp}$ and has the same characteristics.

## 4 The cross-term $C_{\epsilon\|\perp}$

This term expresses the coupling between the aliasing on the mirror, the temporal filtering of the residual parallel phase and the orthogonal phase. For an accurate estimate, it is essential to take into account the temporal aspects of the loop. We develop, in the following, a time series approach of the decomposition of the residual phase into two components ($\|$ and $\perp$), neglecting the filtering of the integration time of the wavefront sensor (WFS) so that the measurement corresponds to an instantaneous phase and not an average phase over the exposure time.

Let us consider a simplified AO system such as described in Fig. 2.

Considering this model, we found an expression for the parallel component of the phase at iteration $i$ as shown in Eq. 12.

$$\epsilon_\|^{\,i} = (1-g)^n \epsilon_\|^{\,i-n} - \sum_{m=1}^{n} g(1-g)^{m-1}\left(\boldsymbol{D}^+ \boldsymbol{D}'_\infty \epsilon_\perp^{\,i-m}\right)$$

$$+ \sum_{m=1}^{n}(1-g)^{m-1}\Delta\Phi_\|^{i-m}, \tag{12}$$

In this equation, $g$ is the loop gain, $\boldsymbol{D}'_\infty$ is the KL interaction matrix in the DM orthogonal space, $\boldsymbol{D}^+$ is the command matrix on the mirror modes and $\Delta\Phi_\| = \Phi^{i-m+1} - \Phi_\|^{i-m}$. $n$ is the number of previous iterations to consider. If $n$ is well chosen, the first term of the equation tends to zero and therefore $\epsilon_\|^{\,i}$ does not depend on $\epsilon_\|$ at previous iterations, but only on $\Phi_\|$ and $\epsilon_\perp$, i.e. $\Phi_{\text{atm}}$.
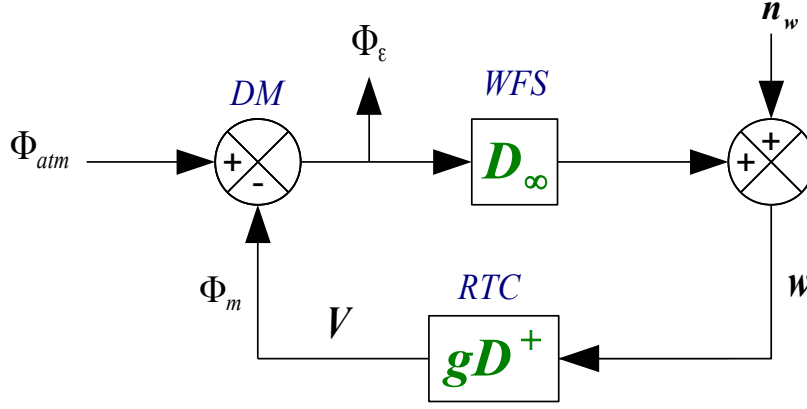
**Fig. 2.** Block diagram of an AO system where the different phases, subsystems and system operators involved are shown. DM, WFS and RTC are respectively the deformable mirror, the wavefront sensor, and real time computer. $\Phi_{\text{atm}}$, $\Phi_{\epsilon}$, $\Phi_{\text{m}}$ are respectively the turbulent atmospheric phase, the residual phase and the mirror phase. $\boldsymbol{n}_w$, $\boldsymbol{w}$ and $\Delta V$ respectively represent the measurement noise vector, the measurement vector and corrective commands sent to the deformable mirror. Finally, $\boldsymbol{D}_\infty$ is the interaction matrix (200 KL to measurements) and $\boldsymbol{D}^+$ the command matrix (measurements to commands).

Finally, the cross-term is given by:

$$\boldsymbol{C}_{\epsilon\perp\|} = <\boldsymbol{\epsilon}_\|\boldsymbol{\epsilon}_\perp{}^{\text{t}}> \simeq \underbrace{(1-g)^n <\boldsymbol{\epsilon}_\|{}^{i-n}\boldsymbol{\epsilon}_\perp{}^{i\text{t}}>}_{0} - \sum_{m=1}^{n} g(1-g)^{m-1}\boldsymbol{D}^+\boldsymbol{D}'_\infty <\boldsymbol{\epsilon}_\perp{}^{i-m}\boldsymbol{\epsilon}_\perp{}^{i\text{t}}>$$

$$+ \sum_{m=1}^{n}(1-g)^{m-1} <\Delta\Phi_\|^{i-m}\boldsymbol{\epsilon}_\perp{}^{i\text{t}}> . \tag{13}$$

We see in Eq. 13 that the cross term only depends on temporal covariance matrices of modes of $\Phi_{\text{atm}}$, i.e. on the Kolmogorov spectrum and the Taylor hypothesis reduces the parameter space to two key parameters $r_0$ and the wind speed $\boldsymbol{v}$. This method to estimate the cross-term have been tested on a Canary-like system [Gendron et al.(2011)] in SCAO mode with a Yao[1] simulation. The parameters of the simulation are presented in Tab. 1. The $\boldsymbol{\epsilon}_\|$ term canot be estimated. Thus we need to chose $n$ so that this term in Eq. 13 tends to zero. The figure 3 shows the precision of the cross term estimate versus $n$.

First, we note that the estimation error (the difference between the exact covariance recovered from the simulation and the estimated covariance with Eq. 13) decreases when $n$ increases since the correlation between the true cross term (obtained by simulation) and the one estimated by recurrence is closer and closer to 1 (Fig. 3).

## 5 Preliminary results

To assess the accuracy on the cross-term estimation, we must minimize the ML criterion. We use a conjugated gradients algorithm using the method of [Thiébaut (2002)]. If the accuracy of the cross term is not sufficient enough, the result of the minimization process is wrong or diverges. Measurements cannot be reproduced by the model given the significance of the cross-term on the measurements.
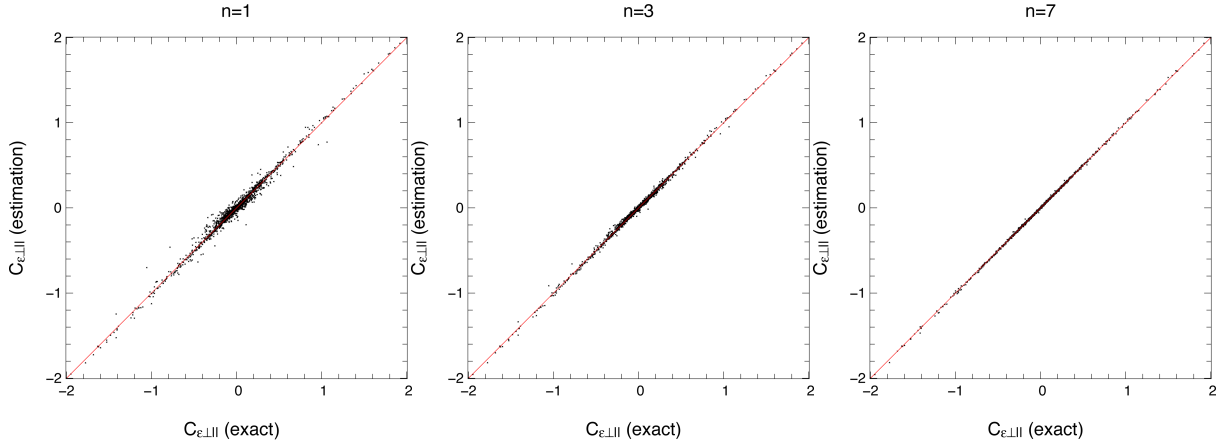
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[1] http://frigaut.github.io/yao/index.html

**Fig. 3.** The covariance $C_{\epsilon\perp\|}$ estimated with the recursive method for different values of $n$ with respect to the exact covariance obtained from the simulation for a system operating at 300 Hz and a gain of $g = 0.5$. The red line represents the function $y = x$.

**Table 1.** Yao parameters in our simulations.

|            | Parameter | Value |
|------------|-----------|-------|
| Simulation | Pupil diameter | 140 px |
| Atmosphere | $r_0$ @ 0.5 $\mu$m | 0.12 m |
|            | Wind speed | 7.5 m s$^{-1}$ |
|            | Layer altitude | 0 m |
| WFS        | Type | SH geometric |
|            | Wavelength | 0.589 $\mu$m |
|            | Number of sub-apertures | 7x7 |
|            | noise | none |
| DM         | Type | KL |
|            | Number of modes | 30 |
|            | Including tip-tilit | yes |
| Telescope  | Diameter | 4.2 m |
|            | Centrale obscuration | 0.25 |
|            | Observed wavelength | 1.65 $\mu$m |
| Loop       | Gain | 0.3–0.5 |
|            | Frequency | 300–150–100 Hz |
|            | Iterations | 5000 |

The figure 4 shows the estimation error on the variance of the modes, i.e. the diagonal of the matrix of Fig. 1, with the ML method for different values of the sampling frequency $\nu$, the loop gain $g$ and the number of used iteration $n$.

For $n = 1$, the variance estimation provided by the ML method is already accurate. Indeed, for most of the modes, we reach an accuracy better than 10% compared to the simulated true variance. Only the first modes (equivalent to the tip-tilt modes) have a significant error and modes 7 and 18. For $n = 3$, the same modes have significant error, but the variance is better estimated than in the case $n = 1$. Finally, for $n > 3$, in our case $n = 5$, the variance is estimated with an error better than 1% with respect to the exact variance for all the modes.

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By varying the frequency of the loop, we reach at the same conclusion. With a loop at 150 Hz or 100 Hz , modes of higher order are estimated with errors greater than 10% for $n = 1$. For $n = 5$, we also reproduce accurately the variance.

We can therefore conclude that the recursive method reaches sufficient accuracy on the cross term to allow a very accurate estimate of the variance of the modes for $n \geq 5$ in this case.

The choice $n$ depends on the gain $g$ . With a lower gain, the function $(1-g)^n$ decreases slowly with $n$. The figure 4 shows as well the result of the minimization for a loop gain $g = 0.3$.

In this case, for $n = 1$, the variance of the error is large, as well as for $n = 3$. Indeed, with $n = 1$ and $g = 0.3$, the term $\epsilon_{\parallel}$ is still 70% of its value and is not negligible. For $n = 3$, $(1 - g)^n = 0.34$ which is important. The variance estimation with more than 1% accuracy is obtained for $n = 9$ (not shown).
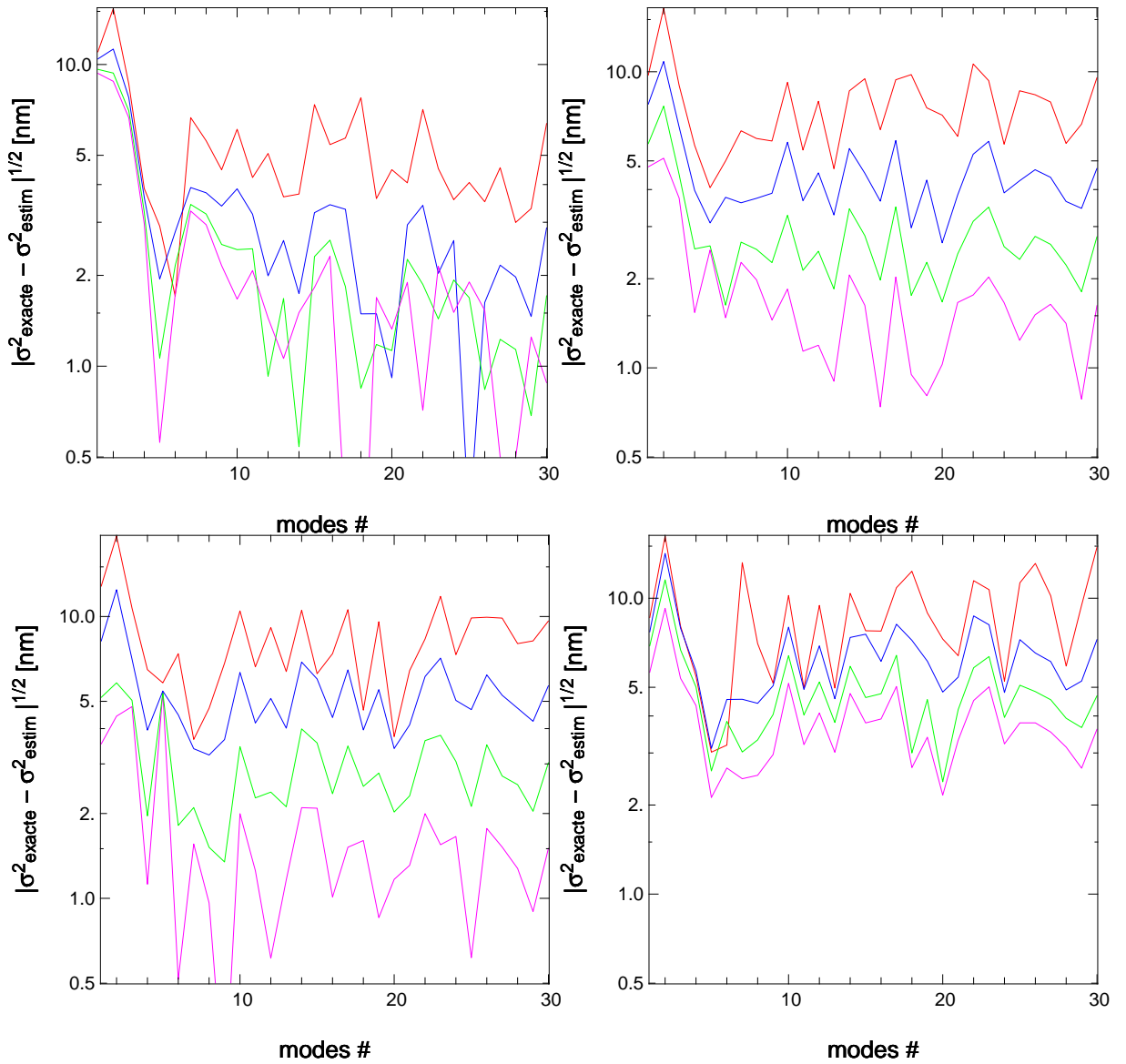


**Fig. 4.** Absolute errors on the estimated variance with respect to the mode for $n = 1$ (red), $n = 3$ (blue), $n = 5$ (green) and $n = 7$ (magenta). Top left, for a simulated system with $g = 0.5$ and a frequency of $\nu = 300$ Hz. Top right, $g = 0.5$ and $\nu = 150$ Hz. Bottom left, $g = 0.5$ and $\nu = 100$ Hz. Bottom right, $g = 0.3$ and $\nu = 300$ Hz.

# 6 Conclusion

We have developed a new method for AO PSF reconstruction allowing to estimate the parallel component of the residual phase based on maximum likelihood approach. This method allows to estimate the the most likely covariance matrix of the modes knowing measurements taking explicitly into account the effect of aliasing, temporal bandwidth and noise. To do this, we need an accurate estimate of the correlation terms between the parallel and the perpendicular components of the residual phase. These terms have a significant effect on the measurements and must be taken into account when estimating the parallel component. We developed an analytical expression for the estimation of these cross terms from the expansion of the atmospheric phase of the KL and relying on Kolmogorov statistics and Taylor hypothesis. The estimation of these terms is done using a recursive method, which takes into account the temporal effects in AO system, which will allow us to express the correlation function with only $r_0$ and the wind speed $v$.

With this estimation, we tested the ML method in a simplified case. We successfully estimated the variance of the modes with an accuracy better than 1% by the selection of the number of iterations to be considered in the recursive expression.

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